When you come back to school, you will be expected to have attempted every problem. These skills are all different tools that you will pull out of your “toolbox” this year to help us solve calculus problems. You will be way behind at the beginning of the year if you haven’t attempted as much as you can! We will spend a few days working together on the items students struggled with. Shortly into the school year, we will test over this material.

The summer assignment is broken down into six tutorials. Each tutorial contains some definitions and some example problems. You should read through all of these, and consult these examples when you have questions. The end of the tutorial will have a section entitled “homework.” It is these six “homework” sections that will be due at the beginning of the school year. All of the material you’ve seen before in previous courses, but possibly adapted in different ways (calculus is the course that puts everything together).

The six tutorials are:

- Tutorial #1 – Linear Functions
- Tutorial #2 – Function Characteristics
- Tutorial #3 – Piecewise Functions
- Tutorial #4 – Exponential and Logarithmic Functions
- Tutorial #5 – Trigonometric Expressions and Equations
- Tutorial #6 – Complex Algebraic Simplifications

All homework assignments should be completed on separate paper. Any graph sketches should include the basic shape of the function and any characteristics (asymptotes, max/min, etc) should be clearly labeled. Round any approximate answers to three decimal places.

Should you lose this assignment, it can be found on the BVSW Homepage in the bottom right corner. If you run into any problems on this assignment over the summer, feel free to e-mail me. My e-mail address is BHawks@bluevalleyk12.org.

I look forward to seeing you in the fall!

Mr. Hawks
Example 1: Consider the line with slope -2 that passes through the point (4, -3).

a. Write the equation of the line in point-slope form.

Point-slope form is \( y = m(x - x_1) + y_1 \), where \( m \) is the slope and \( (x_1, y_1) \) is a point on the line.

For this particular problem then, the equation is \( y = -2(x - 4) - 3 \).

b. Graph the line.

When you graph this line, first plot the point (4, -3) and then use the slope of -2. Here’s what the final graph looks like in a standard viewing window:

c. Find the area between this line and the x-axis from when \( x = -2 \) to \( x = 2 \).

The region in question is a trapezoid as shown in the diagram above. The area of a trapezoid is the average of the parallel sides times the perpendicular side. Since the slant height of the trapezoid runs from (-2, 9) to (2, 1) [We can get these points by subbing -2 and 2 in for \( x \) in the line’s equation], the trapezoid has parallel heights of 9 and 1, and a base of 4. Thus, its area is \( (4) \left( \frac{9 + 1}{2} \right) = (4) \left( \frac{10}{2} \right) = 20 \).
d. Find the equation of the line perpendicular to this line that passes through the point (-5, 2). Leave your answer in point-slope form.

Since the original line has slope $-2$, the perpendicular line has slope $\frac{1}{2}$. Thus, the answer is:

$$y = \frac{1}{2}(x + 5) + 2.$$

e. Find the equation of the line parallel to the line that passes through the point (1, 8). Leave your answer in point-slope form.

Since the original line has slope $-2$, the parallel line has slope $-2$. Thus, the answer is:

$$y = -2(x - 1) + 8.$$

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**Homework problems from tutorial #1:**

1) Consider the line $y = 3x + 2$.
   
   a. Sketch the line.
   
   b. Find the area between the line and the x-axis from when $x = 1$ to $x = 4$.

2) Consider the line $y = 5$.
   
   a. What is the slope of this line?
   
   b. Sketch the line.
   
   c. Find the area between this line and the x-axis from when $x = -2$ to $x = 3$.

3) Consider the line $x = 7$.
   
   a. What is the slope of this line?
   
   b. Sketch the line.

4) Consider the point (-3, -9).
   
   a. Find the equation of the horizontal line that passes through this point.
   
   b. Find the equation of the vertical line that passes through this point.

5) Consider the line with a slope of 4 that passes through the point (-2, 5).
   
   a. Find the equation of this line in point-slope form.
   
   b. Find the equation of a line perpendicular to this line that passes through the point (1, 9). Leave your answer in point-slope form.
Tutorial #2 – Function Characteristics:

Example 1: Let \( f(x) = x^3 - 5x^2 + 3x + 7 \).

(a) Use your calculator to help you draw a careful graph of the function.

![Calculator Graph](image)

The calculator’s graph is included above. Your graph should label the intercepts and extrema. You’ll use your calculator’s “zero” or “max” or “min” tool to find these values. In this case, the \( x \)-intercepts are (\(-.866, 0\)), (\(2.211, 0\)), and (\(3.655, 0\)). [Note: since exact values are not available, we will round to three decimal places. In AP Calculus, **always** round to three decimal places.] The \( y \)-intercept is at (\(0, 7\)). The graph’s maximum point occurs at (\(.333, 7.481\)) and the graph’s minimum point occurs at (\(3, -2\)).

(b) Give the domain of the function.

The domain of this function is all real numbers \( \to (-\infty, \infty) \).

(c) Give the equations of any asymptotes.

Polynomial functions don’t have any asymptotes.
(d) Determine when the function is increasing and decreasing.

When describing where a function is increasing and where a function is decreasing, we use x-coordinates only. Additionally, the AP Calculus curriculum expects you to include endpoints on increasing or decreasing intervals, if these endpoints are in the domain of the function.

In this case, then, we say \( f \) is increasing \((\infty, 0.333] \cup [3, \infty)\). We say that \( f \) is decreasing over the interval \([0.333, 3]\).

(e) Determine where the function has any maximum or minimum values.

The maximum value of the function is 7.481 and it occurs when \( x = .333 \).
The minimum value of the function is -2 and it occurs when \( x = 3 \).

(f) Determine when the function is positive and negative.

The function is positive when the graph is above the x-axis. This is \((-\infty, 2.211) \cup (3.655, \infty)\).
The function is negative when the graph is above the y-axis. This is \((-\infty, -2) \cup (3, \infty)\).

(g) Express the end behavior of \( f \) using limit notation. (Use the statements \( \lim_{x \to \infty} f(x) \) & \( \lim_{x \to -\infty} f(x) \).)

As the \( x \)-values get very positive, the curve goes up, so \( \lim_{x \to \infty} f(x) = \infty \).
As the \( x \)-values get very negative, the curve goes down, so \( \lim_{x \to -\infty} f(x) = -\infty \).

Example 2: Let \( f(x) = \frac{x^2 + 3}{x - 1} \).

(a) Use your calculator to help you draw a careful graph of the function.

The calculator’s graph is included above. Your graph should include the vertical asymptote at \( x = 1 \), the y-intercept at \( (0, -3) \) and clear labeling of the points where maximum and minimum values occur. These points are \((-1, -2) \) and \((3, 6)\).
(b) Give the domain of the function.

The domain of this function is all real numbers except where the vertical asymptote occurs. We would write this as \((-\infty,1) \cup (1,\infty)\).

(c) Give the equations of any asymptotes.

There is a vertical asymptote at \(x = 1\). There is no horizontal asymptote as the degree of the numerator is greater than the degree of the denominator, however there is a slant (oblique) asymptote. To find this asymptote, we divide \(x^2 + 3\) by \(x - 1\) and take the quotient. While we could use long division for this, synthetic division is easier. As seen from the synthetic division below, the slant asymptote is at \(y = x + 1\).

\[
\begin{array}{c|ccc}
1 & 1 & 0 & 3 \\
 & 1 & 1 & \\
\hline
1 & 1 & 4 \\
\end{array}
\]

(d) Determine when the function is increasing and decreasing.

In this problem, we say the function is increasing over the intervals \((-\infty,-1] \cup [3,\infty)\). The function is decreasing over the intervals \([-1,1) \cup (1,3]\). Note that we do not bracket the \(x = 1\) as \(x = 1\) is not in the domain of this function.

(e) Determine where the function has any maximum or minimum values.

The function has a minimum value of 6, which occurs when \(x = 3\). The function has a maximum value of -2, which occurs when \(x = -1\). While this may seem weird, it’s important to remember that these points are merely local extreme values, and not absolute extrema. (We’ll get into this idea more in chapter 4). Additionally, this idea of having a minimum with a larger value than a maximum is possible when the function isn’t continuous for all real numbers (as is the case here).

(f) Determine when the function is positive and negative.

The function is positive when the graph is above the \(x\)-axis. This is \((1,\infty)\). The function is negative when the graph is below the \(x\)-axis. This is from \((-\infty,-1)\).

(g) Express the end behavior of \(f\) using limit notation. (Use the statements \(\lim_{x \to \infty} f(x)\) & \(\lim_{x \to -\infty} f(x)\).)

As \(x\)-values get larger and larger, the rational function continues to go up, so we say that \(\lim_{x \to \infty} f(x) = \infty\). As \(x\)-values get increasingly negative (as we go to the left), the rational function is on its way down. So, we say that \(\lim_{x \to -\infty} f(x) = -\infty\).
Homework problems from tutorial #2:

For questions 1-5, complete the following steps:

(a) Use your calculator to help you draw a sketch of the function.
(b) Give the domain of the function.
(c) Give the equations of any asymptotes. Mark these on your sketch.
(d) Determine when the function is increasing and decreasing.
(e) Determine where the function has any maximum or minimum values.
(f) Determine when the function is positive and negative.
(g) Express the end behavior of f using limit notation. (Use the statements \( \lim_{x \to \infty} f(x) \) & \( \lim_{x \to -\infty} f(x) \).)

1. \( f(x) = x^3 - x \).
2. \( g(x) = x^2 e^{-x} \).
3. \( h(x) = \frac{2x}{x+7} \).
4. \( f(x) = -\sqrt{25-(x-2)^2} \).

Recall that the formula for a circle centered at (0,0) is \( x^2 + y^2 = r^2 \). If we wanted to graph this function on our calculators, we’d have to solve for \( y \). Solving this equation for \( y \) means we need to subtract \( x^2 \) and then take the square root. (We can’t forget the plus or minus when we take the square root!) This gives us \( y = \pm \sqrt{r^2 - x^2} \). The equation \( y = \sqrt{r^2 - x^2} \) represents the top-half of a circle (or a positive semi-circle) and the equation \( y = -\sqrt{r^2 - x^2} \) represents the bottom-half of a circle (or a negative semi-circle).

5. Consider the equation \( y = \sqrt{36-x^2} \).
   a. What is the radius of this semi-circle?
   b. Graph the semi-circle.
   c. Find the area between the semi-circle and the x-axis.
   d. Where is the function increasing and decreasing?
   e. What are the maximum and minimum values of this function?

Problems from Tutorial #2 continue on the next page:
In Calculus, we will frequently use tables of values to learn more information about functions. In the table you are given below, you are given six selected x-values, and corresponding values of f(x) and g(x). For example, if you were to look at the column where x = 0, you would see that f(x) = -2 and g(x) = 3. This means that the point (0, -2) is on the graph of f(x) and the point (0, 3) is on the graph of g(x). It also means that f(0) = -2 and g(0) = 3. Use this information to help answer the questions below.

Questions like 8-10 are function composition questions – not included in the tutorial, but if you’ve forgotten, try google searching “Composition of Functions”.

Suppose f(x) and g(x) are continuous functions with domains of [0, 5].

<table>
<thead>
<tr>
<th>x</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>f(x)</td>
<td>-2</td>
<td>3</td>
<td>4</td>
<td>6</td>
<td>8</td>
<td>11</td>
</tr>
<tr>
<td>g(x)</td>
<td>3</td>
<td>5</td>
<td>-6</td>
<td>1</td>
<td>0</td>
<td>-4</td>
</tr>
</tbody>
</table>

Use the table to answer questions 6 – 13.

6. Compute \( f(1) + g(1) \).

7. Compute \( f(1)g(2) + f(2)g(1) \).

8. Compute \( f(g(0)) \).

9. Compute \( g(f(1)) \).

10. Compute \( f(3g(4)) \).

11. Write the equation of the line, in point-slope form, with slope 7 at the point corresponding to g(2).

12. Write the equation of the line, in point-slope form, perpendicular to the line in \#15, and passing through the same point.

13. Compute \( g(1) \cdot f(1) \), and explain why this answer is different from your answer to \#12.
Tutorial #3 – Piecewise Functions

In Pre-Calc, you looked at the concept of graphing specific parts of functions. This particular tool is very helpful to us in the study of calculus. Our focus here will be on graphing piecewise functions, and evaluating them at specific points.

Example 1: Graph the function: \( f(x) = \begin{cases} 2x - 3, & \text{if } x > 1 \\ \sin(\pi x), & \text{if } x \leq 1 \end{cases} \)

In Pre-Calculus, the way that we graphed piecewise functions is that we graphed both of the original functions and then kept the part of the domain that was requested in the function. Your procedure for solving this problem would be to graph \(2x - 3\), and only keep the part of the line to the right of \(x = 1\) (i.e. where \(x > 1\)), and then to graph the sinusoid and only keep the part of the sinusoid to the left of \(x = 1\).

In Calculus, there are a few things that you’ll be interested with regards to the piecewise function. First, you’ll need to be able to describe the behavior around the function’s discontinuity (\(x = 1\)). Second, you’ll need to be able to describe the type of discontinuity (jump).

A jump discontinuity has a gap. See the piecewise graphed above at \(x = 1\).

A removable discontinuity looks like a point has been picked out of a function and either tossed off of the graph or placed somewhere else on the same vertical line where the removal occurred. Pictures of removable discontinuities:
Example 2: Consider \( f(x) = \begin{cases} 2x - 3, & \text{if } x < 0 \\ x^2 + x - 2, & \text{if } 0 \leq x \end{cases} \).

(a) Find \( f\left(-\frac{3}{4}\right), f(0), f(1) \) and \( f(2) \).

To compute \( f\left(-\frac{3}{4}\right) \), we are being asked for the \( y \)-value of the function when \( x = -3/4 \), so it’s important to determine which piece of the piecewise function we’re supposed to use here. In this case, since \(-3/4\) is less than 0, we are using the “\(2x - 3\)” piece. So,
\[
f\left(-\frac{3}{4}\right) = 2\left(-\frac{3}{4}\right) - 3 = -\frac{3}{2} - 3 = -\frac{9}{2}.
\]

When \( x = 0 \), the “\(\sin(\pi x)\)” piece is being used, as the domain of this piece includes zero, while the domain of the other piece does not. \( \sin(\pi(0)) = \sin(0) = 0 \).

When \( x = 1 \), \( \sin(\pi(1)) = \sin(\pi) = 0 \).

When \( x = 2 \), the function is not defined, as \( x = 2 \) does not fit in the domain of either piece of this function.

(b) Find all values of \( x \) for which \( f(x) = 0 \).

This is a significantly harder question than (a), because this question is asking “for what values of \( x \) does the function have \( y \)-values of 0?” Since either piece of the function could theoretically have \( y \)-values of 0 (remember, the piecewise function restricts what \( x \)-values go with which functions, not which \( y \)-values go with which functions), we have to set each piece equal to 0 and solve both.

Set \( 2x - 3 = 0 \). This leads to \( 2x = 3 \Rightarrow x = \frac{3}{2} \). So, the linear piece has a \( y \)-value of 0 when the \( x \)-value is \( \frac{3}{2} \). However, the domain of this piece is \( x < 0 \), so \( x = \frac{3}{2} \) is not in our domain, and is therefore, not a solution to our equation.

Set \( x^2 + x - 2 = 0 \). We can solve this by factoring. Factor this into \((x + 2)(x - 1) = 0\). This gives us solutions of \( x = -2 \) and \( x = 1 \); however, the domain of this function is only \( x \geq 0 \), so only \( x = 1 \) fits our domain. The only solution to the entire equation is \( x = 1 \).
Homework problems from tutorial #3:

1. Consider the piecewise function, \( f(x) = \begin{cases} 
3x & \text{if } x > 0 \\
5 & \text{if } x = 0 \\
x & \text{if } x < 0
\end{cases} \).
   
   (a) Draw a graph of \( f(x) \).
   (b) Does the function have a jump or a removable discontinuity at \( x = 0 \)? Explain.
   (c) Find \( f(0) \).
   (d) Find \( f(10) \).
   (e) Find the value(s) of \( x \) for which \( f(x) = 10 \).
   (f) Calculate \( \lim_{x \to \infty} f(x) \).
   (g) Calculate \( \lim_{x \to -\infty} f(x) \).

2. Consider the piecewise function, \( f(x) = \begin{cases} 
\frac{1}{x} & \text{if } x \geq 1 \\
e^x - 2 & \text{if } x < 1
\end{cases} \).
   
   (a) Draw a graph of \( f(x) \).
   (b) Does the function have a jump or a removable discontinuity at \( x = 1 \)? Explain.
   (c) Find \( f(1) \).
   (d) Find \( f(-3) \).
   (e) Find the value(s) of \( x \) for which \( f(x) = -3 \).
   (f) Calculate \( \lim_{x \to \infty} f(x) \).
   (g) Calculate \( \lim_{x \to -\infty} f(x) \).

3. Draw an example of a jump discontinuity and removable discontinuity; then, explain the difference between the two.
Tutorial #4 – Exponential and Logarithmic Functions

Concept 1 – Exponential Functions

In Calculus, we will deal a little bit with exponential functions. An exponential function is defined by the equation, \( f(x) = a^x \). In this definition, \( a \) must be positive and not equal to 1, while \( x \) must be a real number, not equal to zero.

Example 1: Let \( f(x) = 16^x \). Evaluate \( f(0), f(2), f(1/2), f(-1), \) and \( f(-3/4) \).

Solutions:

\[
f(0) = 16^0 = 1. \text{ (Anything to the zero power is 1.)}
\]

\[
f(2) = 16^2 = 256.
\]

\[
f\left(\frac{1}{2}\right) = 16^{\frac{1}{2}} = \sqrt{16} = 4.
\]

(Remember that anything to the \( \frac{1}{2} \) power is the same as finding the square root.)

\[
f(-1) = 16^{-1} = \frac{1}{16} = \frac{1}{16}.
\]

(Any negative exponent can be made positive by moving the base and the power to the other part of the fraction.)

\[
f\left(-\frac{3}{4}\right) = 16^{-3/4} = \frac{1}{16^{3/4}} = \frac{1}{\left(\sqrt[4]{16}\right)^3} = \frac{1}{2^3} = \frac{1}{8}.
\]

Example 2: Let \( f(x) = 2^x \), and \( g(x) = 2^{x+1} \).

(a) Find \( g(f(1)) \).

Solution: Since you want to calculate \( g(f(1)) \), you need to find \( f(1) \) first. Once you find \( f(1) \), substitute that result into \( g \). \( f(1) = 2^1 = 2 \). \( g(2) = 2^{2+1} = 2^3 = 8 \).

(b) Find \( \left(\frac{g}{f}\right)(x) \).

Solution: \( \left(\frac{g}{f}\right)(x) = \frac{2^{x+1}}{2^x} = 2^{x+1-x} = 2^1 = 2 \). Since you are dividing two exponential expressions with like bases, you may subtract the exponents.

Example 3: Solve the exponential equations:
(a) \( 2^x = 32. \)

**Solution:** This exponential equation can be solved easily by equating bases. Since both the left and right hand side of the equations can be expressed as powers of 2, rewrite 32 as \( 2^5. \) Thus, you can rewrite the equation \( 2^x = 2^5. \) Thus, if you try and match both sides of the equation \( x = 5. \)

(b) \( 4^x = 8 \)

**Solution:** You can rewrite both sides of the equation as powers of 2. Rewrite \( 4^x \) as \( 2^{2x}. \) Rewrite 8 as \( 2^3. \) Then, you’d have \( 2^{2x} = 2^3. \) Then \( 2x = 3, \) so \( x = 3/2. \)

(c) \( 8^{2x+5} = \frac{1}{16} \)

**Solution:** Again, rewrite both sides as a power of 2. \( 2^{3(2x+5)} = \frac{1}{2^4}, \) then \( 6x + 15 = -4, \) \( x = \frac{-19}{6}. \)

The main motivation for logarithmic functions stem from problems similar to what you saw in example 3. In example 3, we were able to solve all of our problems by simply equating bases. However, what happens in the situation where bases can’t be equated?

For example, what happens if we want to solve \( 5^x = 17? \) Clearly, we can’t rewrite 17 as a power of 5. When mathematicians encountered this problem, they knew that getting the exponent out of the variable was a must. The way they removed the exponent from the variable was by taking either the logarithm or natural logarithm of both sides (it doesn’t matter which one you use as long as you use it on both sides of the equation.)

**Logarithmic functions** are the inverse functions of exponential functions. In other words, if \( y = a^x, \) then \( \log_a(y) = x. \)

**Example 4:** Rewrite the following exponential equations in logarithmic form.

a) \( 5^{-2} = \frac{1}{25} \)

**Solution:** \( \frac{1}{25} = 5^{-2} \Rightarrow \log_5\left(\frac{1}{25}\right) = -2. \)

b) \( a^c = b \)

**Solution:** \( b = a^c \Rightarrow \log_a(b) = c. \)

**Properties of logarithms:**

\[
\begin{align*}
\ln(a^b) &= b \ln a \\
\ln(a) + \ln(b) &= \ln(ab) \\
\ln(a) - \ln(b) &= \ln\left(\frac{a}{b}\right) \\
\ln(e) &= 1 \\
\ln(1) &= 0 \\
\ln(e^{f(x)}) &= f(x) \\
\frac{\ln b}{\ln a} &= \frac{\log b}{\log a} = \log_a b
\end{align*}
\]
Example 5: Rewrite the following expressions using properties of logarithms.

a) \( \ln \left( \frac{e}{e^x} \right) = \)

Solution: \( \ln \left( \frac{e}{e^x} \right) = \ln \left( \frac{e^1}{e^x} \right) = \ln (e^{1-x}) = 1 - x \).

b) \( \ln (e^{1-2x}) + \ln(e) = \)

Solution: \( \ln (e^{1-2x}) + \ln(e^1) = \ln(e^{1-2x+1}) = \ln(e^{2-2x}) = 2 - 2x \)

Example 6: Solve the following equations:

a) \( 5^{2x-3} = 72 \)

Solution: Start by taking the logarithm or natural log of both sides. Then, we will have \( \ln(5^{2x-3}) = \ln(72) \). Now, use a property of logarithms to pull the exponent out front of the left-hand expression. So, then we’ll have \( (2x-3)\ln 5 = \ln 72 \). Then, solve for \( x \), and you’ll get

\[
2x - 3 = \frac{\ln 72}{\ln 5} \Rightarrow 2x = 3 + \frac{\ln 72}{\ln 5} \Rightarrow x = \frac{3 + \frac{\ln 72}{\ln 5}}{2}.
\]

b) \( \ln(x + 4) - \ln(5) = 3 \)

Solution: First use properties of logs to rewrite \( \ln(x + 4) - \ln(5) = 3 \) as \( \ln \left( \frac{x + 4}{5} \right) = 3 \). Then, we want to rewrite to eliminate the natural log. Thus, \( e^3 = \frac{x + 4}{5} \). If we want to solve this for \( x \), we’d need to start by multiplying through by the denominator \( 5e^3 = x + 4 \Rightarrow 5 = x - 4 \).

Example 7: Solve for \( y \): \( -\ln |y + 3| = 2x - \ln 3 \).

Solution: This equation is tricky to solve, but we are first going to need to get rid of the negative on the left-hand side of the equation. So, let’s divide both sides through by -1. This gives us \( \ln |y + 3| = -2x + \ln 3 \). Then, we need to undo the natural log on the left hand side of the equation. This means we now have the statement \( |y + 3| = e^{-2x + \ln 3} \). Next, we are going to make a funny simplification. Since we have added exponents in the right hand side of the equation, we can rewrite that side to have multiplied bases. In other words, \( e^{-2x + \ln 3} = e^{-2x} e^{\ln 3} \). This is useful because we know that \( e^{\ln 3} = 3 \). So, \( e^{-2x + \ln 3} = 3e^{-2x} \). This means our overall equation is \( |y + 3| = 3e^{-2x} \). Next, we need to undo the absolute value bars. We do this by recognizing that the right hand side of the equation may be either positive or negative, so \( y + 3 = \pm 3e^{-2x} \). Isolating \( y \) gives us \( y = -3 \pm 3e^{-2x} \).
Homework from Tutorial #4:

1. Explain why $f(x) = e^x$ is an exponential function while $f(x) = x^2$ is not an exponential function.

2. Explain why $f(x) = a^x$ is not an exponential function in the situations where $a = 1$ or where $x = 0$. What kind of function would $f(x)$ be in these situations?

3. Let $f(x) = 8^x$. Evaluate $f(0), f(2), f(-2),$ and $f(-5/3)$.

4. Evaluate $\log_3\left(\frac{1}{27}\right)$.

5. Simplify the following expressions using properties of logarithms:
   (a) $\ln(e^{-8}) - \ln(e^{-11})$
   (b) $\log_2(2^a)$
   (c) $\ln(e) - \ln(1)$
   (d) $\ln\left(\frac{1}{e^y}\right)$

6. Solve the following equations for $y$:
   (a) $\ln|y| = \frac{x^2}{2} + \ln 3$
   (b) $-e^{-y} = e^x + 5$

7. Find $f(1) - f(0)$ if $f(x) = \frac{e^{-4x}}{-4}$.

8. Evaluate when $x = 0$: $\frac{1}{x + 4 + e^{-3x}(1 - 3e^{-3x})}$
Tutorial #5 – Trigonometric Expressions and Equations

Example 1: Use the unit circle to evaluate the following expressions:

(a) \( \cos \left( \frac{5\pi}{3} \right) = \frac{1}{2} \).

(b) \( \cot \left( \frac{7\pi}{6} \right) = \frac{\cos \left( \frac{7\pi}{6} \right)}{\sin \left( \frac{7\pi}{6} \right)} = \frac{-\sqrt{3}/2}{-1/2} = \sqrt{3} \).

(c) \( \csc \left( -\frac{19\pi}{3} \right) = \csc \left( -\frac{13\pi}{3} \right) = \csc \left( -\frac{7\pi}{3} \right) = \csc \left( \frac{\pi}{3} \right) = \csc \left( \frac{5\pi}{3} \right) = \frac{1}{\sqrt{3}/2} = \frac{2}{\sqrt{3}} \).

(d) \( \tan(12\pi) = \tan(10\pi) = \tan(8\pi) = \tan(6\pi) = \tan(4\pi) = \tan(2\pi) = \tan(0) = \frac{\sin(0)}{\cos(0)} = 0 = 0 \).

Example 2: Evaluate the following inverse trigonometric functions.

(Remember that inverse trigonometric functions use ratio as inputs, and output angles.)

(a) \( \sin^{-1} \left( \frac{\sqrt{3}}{2} \right) = \frac{\pi}{3} \).

(Keep in mind that the range of arcsine is from \( \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right] \).) The angle in the range that has a sine of \( \sqrt{3}/2 \) is 60° or \( \pi/3 \).

(b) \( \tan^{-1} \left( \sqrt{3} \right) = \frac{\pi}{3} \).

(Keep in mind that the range of arctangent is from \( \left( -\frac{\pi}{2}, \frac{\pi}{2} \right) \).) The angle in the range that has a tangent of \( \sqrt{3} \) is the angle that has a sine of \( \sqrt{3}/2 \) and a cosine of 1/2.

(c) \( \cos^{-1} \left( -\frac{1}{2} \right) = \frac{2\pi}{3} \).

(Keep in mind that the range of arccosine is from \( [0,\pi] \).) The angle in this range that has a cosine of -1/2 is \( 2\pi/3 \).
(d) \( \sin^{-1}\left( -\frac{\sqrt{2}}{2} \right) = -\frac{\pi}{4} \)

(Keep in mind that the range of arcsine is from \( \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right] \).) The angle in this range that has a sine of \( -\frac{\sqrt{2}}{2} \) is \( -\pi/4 \). Many of you will want to use either \( \frac{5\pi}{4} \) or \( \frac{7\pi}{4} \) as the answer here. However, both of these are wrong, because they don’t fit into the range.

Homework from Tutorial #5:

1. Fill in the blank unit circle – label each angle in degrees and radians, then label each ordered pair. You may use this one if you like, or print off one from google. You are required to have the unit circle memorized!
2. Evaluate the following without the use of a calculator.

(a) \( \cos \left( \frac{7\pi}{6} \right) = \) 
(b) \( \cot \left( \frac{2\pi}{3} \right) = \) 
(c) \( \csc \left( \frac{3\pi}{4} \right) = \)

(d) \( \tan(\pi) = \) 
(e) \( \cos^{-1} \left( \frac{\sqrt{3}}{2} \right) = \) 
(f) \( \tan^{-1}(1) = \)

(g) \( \sin^{-1} \left( -\frac{1}{2} \right) = \) 
(h) \( \cos^{-1} \left( -\frac{\sqrt{2}}{2} \right) = \)

3. Explain the difference between \( y = \csc x \) and \( y = \sin^{-1} x \).

3. Evaluate the following expressions, if \( f(x) = 3\sin(3x) \).

(a) Find \( f \left( \frac{\pi}{2} \right) \). 
(b) Find \( f \left( \frac{3\pi}{4} \right) \). 
(c) Find \( f \left( \frac{-\pi}{18} \right) \).
Tutorial #6 – Complex Algebraic Simplifications:

Example 1: Determine what \( \frac{f(x)g(x) - h(x)j(x)}{f^2(x)} \) is if \( f(x) = x^2 - 2x + 1 \), \( g(x) = -e^{-x} \), \( h(x) = e^{-x} \), \( j(x) = 2x + 2 \).

Solution:
By substitution, we immediately have \( \frac{f(x)g(x) - h(x)j(x)}{f^2(x)} = \frac{(x^2 - 2x + 1)(-e^{-x}) - (e^{-x})(2x + 2)}{(x^2 - 2x + 1)^2} \).
After distributing, we have \( \frac{-x^2e^{-x} + 2xe^{-x} - e^{-x} - 2xe^{-x} - 2e^{-x}}{(x^2 - 2x + 1)^2} \).
Combining like terms and factoring the numerator gives us \( \frac{-x^2e^{-x} - 3e^{-x}}{(x^2 - 2x + 1)^2} = \frac{-e^{-x}(x^2 + 3)}{(x^2 - 2x + 1)^2} \).

Example 2: Solve: \( x^3e^{3x} - 3xe^{3x} = 0 \).

Solution: Start by factoring the common factor out of each term. Some of you will notice that the common factor here is \( xe^{3x} \). However, the process is doable even if you only recognize \( e^{3x} \) as the common factor. Factoring out \( e^{3x} \) gives us \( e^{3x}(x^3 - 3x) = 0 \). Setting each term equal to zero, makes \( x^3 - 3x = 0 \), which we can solve by factoring out an \( x \), yielding \( x(x^2 - 3) = 0 \). So, we have solutions at \( x = 0, x = \sqrt{3}, x = -\sqrt{3} \). Setting \( e^{3x} = 0 \) produces an interesting case, as anything raised to a power won't ever equal zero!! So our only solutions are \( x = 0, x = \sqrt{3}, x = -\sqrt{3} \).

Comment: You will make very heavy use of exponent properties in this section. A review of these properties are listed here.

Negative Exponent Property: \( a^{-n} = \frac{1}{a^n} \).
Raising a quantity to a negative exponent is equivalent to writing one over the quantity with a positive exponent.

Rational Exponents Property: \( a^{m/n} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m \).
Raising a quantity to a rational exponent is the same as raising the quantity to the power of the rational exponent’s numerator and then taking the root of that quantity that corresponds to the number of the denominator.

Example 3: Rewrite \( \frac{5r}{\sqrt{(r+1)^2}} \) so that no variables are in the denominator and no radicals are in your answer.

Solution: First rewrite the denominator as \( (r+1)^{2/2} \). So, we have \( \frac{5r}{(r+1)^{2/2}} \). Then to move that quantity out of the denominator, we need to rewrite it with a negative exponent, and this is our final answer: \( \frac{5r}{(r+1)^{2/2}} = 5r(r+1)^{-2/2} \).
Homework from Tutorial #6:

1) Determine what \( \frac{f(x)g(x) - h(x)j(x)}{f^2(x)} \) is. Simplify your answer as much as possible. [See ex 1]

   a) Let \( f(x) = 2x + 5 \), \( h(x) = x^2 - 3x \), \( g(x) = 2x - 3 \), \( j(x) = 2 \).
   b) Let \( f(x) = x^3 \), \( h(x) = 3x^2 \), \( j(x) = e^{-2x} \), \( g(x) = -2e^{-2x} \).

2) Solve the equation employing techniques you know (factoring, common denominators, etc.):

   a) \( 2xe^{-x} - x^2e^{-x} = 0 \)
   b) \( 5x^4e^{5x} + 4x^3e^{5x} = 0 \)
   c) \( x^2 - 2x^{-1} = 0 \)

3) Simplify each expression using the properties of exponents.

   a) \( (4x^3)(5x^7) \)
   b) \( (12x^4y^3)^2 \)
   c) \( \frac{13a^5b^{-2}z^0}{39ab^{-3}} \)

4) Rewrite the radical expressions using exponents. Your answers will contain neither radicals nor fractions. Your answers, however, may contain negative exponents. (In fact, some of them should ☹️)

   a) \( \sqrt[5]{x^2} \)
   b) \( \frac{7}{\sqrt[3]{x^5}} \)
   c) \( \sqrt[3]{3x} \)
   d) \( \sqrt[9]{\frac{9}{x+1}} \)
   e) \( \frac{2}{\sqrt[3]{3-x}} \)

5) Rewrite each radical as something to a rational exponent (rational exponents are just fractional exponents), then use properties of exponents to combine like terms:

   a) \( y^2\sqrt{y} \)
   b) \( \frac{\sqrt{z}}{z\sqrt{z}} \)
   c) \( x\sqrt{x} \)

6) Rewrite the following expressions so that no variables are in the denominator.

   a) \( \frac{5}{x^7} \)
   b) \( \frac{1}{8x^4} \)
   c) \( \frac{3}{(x+2)^2} \)
   d) \( \frac{2x}{e^x} \)

7) Solve for \( y \):

   a) \( -y^{-3} = x^2 - \frac{5}{8} \)
   b) \( \frac{y^{1/2}}{1/2} = \frac{x^2}{2} + 5x - 3 \)

8) Calculate the value of \( C \) (without using a calculator) if \( -3(2)^{-2} = 8^{2/3} + C \)